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Research note

Analytical approach to dynamic and vibration analysis of a spherical ball under contact stress

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Abstract This paper presents a nonlinear model to illustrate the effect of contact stress in the vibration behavior of mechanical components, especially rotating systems. The problem is considered in the case of the vertical vibration of a sphere on a plate. The Hertzian contact theory is used to obtain the relationship between contact force and the deflection of the mass center of the sphere. Modeling the system by a mass and a nonlinear spring, the vibration equation of the mass center of the sphere is derived. The method of Lindstedt–Poincaré is implemented to solve the equation of motion, and obtain vibration characteristics under a compressive preload. The dependency of frequency on several parameters, such as initial applied force, initial amplitude of oscillation and the diameter of the sphere, is distinguished. Results show that increasing the initial applied force or the diameter of the ball raises the frequency, while increasing oscillation amplitude has an inverse effect. Finally, the accuracy and convergence of the solution are illustrated by comparison between different orders of approximation. Also, results are in good agreement with those extracted from numerical modeling.

1. Introduction

Vibration analysis is highly important in mechanical systems, especially in rotary machines. In order to avoid resonance, the working frequency must be kept away from the natural frequency of the system.

There are sources that affect the vibration response of rotary equipment, for example, mass unbalancing, clearance, coupling misalignment, surface waviness and etc. In order to evaluate vibration characteristics, all effective parameters must be considered, carefully.

Many researchers have studied these parameters and their influence on the dynamic behavior of rotary systems. Yamamoto [1] investigated the effect of bearing radial clearance on the vibration characteristics of vertical rotors supported by ball bearings. Gustafson and Tallian [2] studied waviness influence and observed that lower order ring waviness affects the amplitude of the vibrations at the ball passage frequency.

Meyer et al. [3] reached similar results, theoretically, in the case of radial ball bearings with a linear modeling of the spring characteristics of the balls. Ehrich and O'Connor [4,5] studied the effect of bearing clearances on the dynamic behavior of rotors. The nonlinear behavior of a rotor-bearing system was studied by Choi and Noah [6], using the Harmonic Balance Method with the discrete Fourier transform procedure. Tiwari et al. [7] studied the effect of the radial internal clearance of the ball bearing on the dynamic response of the rotor. Their numerical and experimental results showed instability and chaos in the dynamic response of the rotor.

Harsha et al. [8] developed an analytical model to predict nonlinear dynamic responses in a rotor bearing system, due to surface waviness. Contacts between the rolling elements and the races were considered as nonlinear springs, whose stiffness was obtained using the Hertzian contact theory. Nataraj and Harsha [9] investigated the nonlinear dynamic behavior of an unbalanced rotor–bearing system, due to cage run out. Sinou [10] studied the nonlinear response of a flexible rotor supported by ball bearings, due to unbalanced force.

Contact stress is the other important parameter in the vibration analysis of systems. There are many industrial

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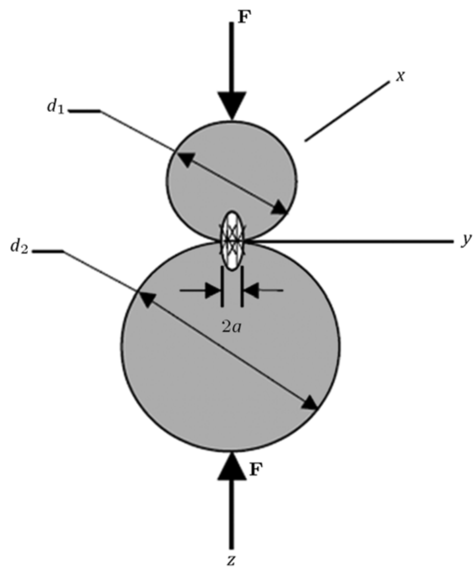

Figure 1: Two spheres held in contact by means of the compressive force [21].

applications in which a rotating component is exposed to a contact force, such as bearings, gear teeth and rolling tires on the roads. Also, it is associated with some very important railway engineering problems, such as wheel flats or other wheel tread irregularities (out of round wheels) [11,12], rail corrugation [13], rail head defects [14] and rolling noise [15,16]. Yan and Fischer [17] analyzed the contact problem in wheel-rail applications. They verified the applicability of the Hertzian contact theory to rail-wheel contact problems, comparing their results with those obtained from three-dimensional finite element models.

Nielsen [18] presented a nonlinear wear model for the contact of a cylinder rolling over a periodically varying surface. His model was a pure contact one and the dynamics of the cylinder and surface were neglected. Baeza et al. [19] studied the dynamic response caused by a geometric irregularity (in rails or wheels), by means of Hertzian and non-Hertzian contact models. El-Sayed [20] obtained the stiffness of bearings using the Hertzian contact model. He also calculated the total deflection of inner and outer races due to an applied force.

Although the effect of contact forces on the vibration behavior of rotary components has been known for many years, a fully satisfactory explanation of the phenomenon has not yet been found.

The objective of this study is to develop an analytical model for the vibration analysis of a spherical ball on a plate under a compressive force. The Hertzian contact theory is used for obtaining the force–displacement relation. Also, the Lindstedt–Poincaré method is implemented to evaluate the natural frequency and time response, analytically.

2. Theory and formulation

2.1. Stress and strain due to compressive load

Figure 1 shows two spheres held in contact by means of the compressive force, F. In order to model the contact between the ball and the plate, the Hertzian contact theory is used. In this theory, principal stresses in x, y and z directions are introduced as [21]:

$$\sigma_z = \frac{-p_{\max}}{1+\zeta_a^2}, \tag{1}$$

$$\sigma_x = \sigma_y = -p_{\max}\left[\left[1-|\zeta_a|\tan^{-1}\left(\frac{1}{|\zeta_a|}\right)\right](1+\nu) - \frac{1}{2\left(1+\zeta_a^2\right)}\right]. \tag{2}$$

$p_{\max}$ and ζ_a appearing in Eqs. (1) and (2) are defined as:

$$p_{\max} = \frac{3F}{2\pi a^2}, \tag{3}$$

$$\zeta_a = \frac{z}{a}, \tag{4}$$

in which:

$$a = K_a \sqrt[3]{F}, \tag{5}$$

$$K_a = \left[\frac{3}{8}\frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{(1/d_1)+(1/d_2)}\right]^{1/3}. \tag{6}$$

In the above equations, E_i's, d_i's and ν_i's are moduli of elasticity, diameters and Poisson ratios of the spheres, respectively, and F is the compressive force. Using stress components obtained from Eqs. (1) and (2), strain components can be evaluated as:

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left(\sigma_x - \nu\left(\sigma_y+\sigma_z\right)\right) \\ \varepsilon_y = \dfrac{1}{E}\left(\sigma_y - \nu\left(\sigma_x+\sigma_z\right)\right) \\ \varepsilon_z = \dfrac{1}{E}\left(\sigma_z - \nu\left(\sigma_x+\sigma_y\right)\right). \end{cases} \tag{7}$$

Integrating the z component of the strain tensor through the radius of the sphere, displacement of the center of the sphere, due to the compressive force, can be calculated as:

$$\delta = \left|\int_0^R \varepsilon_z.dz\right| = \frac{p_{\max}a}{E}\int_0^{d_1/(2a)}\left(\frac{1}{1+\zeta_a^2}\right)d\zeta_a + \frac{2\nu p_{\max}a}{E}\int_0^{d_1/(2a)}\left(\left[-1+|\zeta_a|\tan^{-1}\left(\frac{1}{|\zeta_a|}\right)\right] \times (1+\nu) + \frac{1}{2\left(1+\zeta_a^2\right)}\right)d\zeta_a. \tag{8}$$

Substituting Eq. (4) into Eq. (8), we have:

$$\delta = \frac{p_{\max}}{E}a.C = \frac{3F}{2\pi aE}C, \tag{9}$$

in which C is a function of the geometric and mechanical properties of the system, and can be evaluated from Eq. (8). It is noted that the numeric value of C is obtained after determining the sample's dimensions.

From Eqs. (5) and (9), the relation between force and displacement is obtained as:

$$F = K'\delta^{3/2}, \tag{10}$$

in which K' is introduced as follows:

$$K' = \left(\frac{2\pi EK_a}{3C}\right)^{3/2}. \tag{11}$$

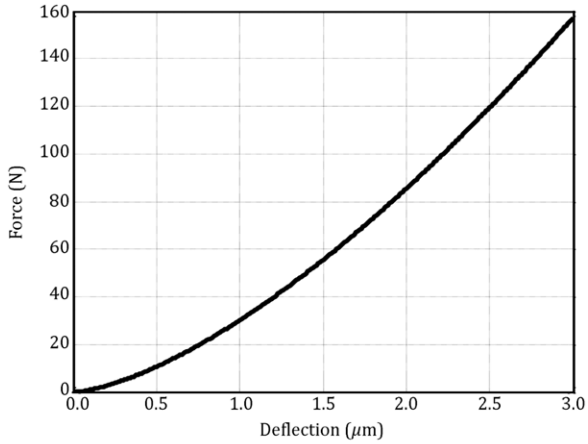

Figure 2: Contact force vs. displacement of the center of the sphere ($K' = 3.02e10$).

2.2. Vibration modeling of the system

It can be realized from Eq. (10) and Figure 2 that the relationship between force and deflection is nonlinear. Modeling the system with a mass and a nonlinear spring, the vibration equation is obtained as:

$$m\ddot{x} + K'x^{1.5} = F, \tag{12}$$

in which m is the equivalent mass and x is the vertical displacement of the center of the sphere relative to the outer surface of the other sphere, respectively. We can rewrite equation (12) in the form of:

$$\ddot{x} + \lambda^2 x^{1.5} = F/m, \tag{13}$$

where:

$$\lambda^2 = \frac{1}{m}\left(\frac{2\pi E K_a}{3C}\right)^{3/2}. \tag{14}$$

It is obvious that by applying the compressive force, the static equilibrium position is displaced from $x = 0$ to $x = x^*$.

In order to solve Eq. (13), we first obtain Taylor expansion of $x^{1.5}$ about x^*:

$$\begin{aligned} x^{1.5} &= (x^*)^{1.5} + \frac{3}{2\times 1!}(x^*)^{0.5}z + \frac{3\times 1}{2^2\times 2!}(x^*)^{-0.5}z^2 \\ &+ \frac{3\times 1\times(-1)}{2^3\times 3!}(x^*)^{-1.5}z^3 \\ &+ \frac{3\times 1\times(-1)\times(-3)}{2^4\times 4!}(x^*)^{-2.5}z^4 \\ &+ \cdots + \frac{3(-1)^n(2n-5)!(x^*)^{1.5-n}}{2^{2n-3}n!(n-3)!}z^n = \sum_{n=0}^{\infty}\alpha_n z^n. \end{aligned} \tag{15}$$

In the above equation, it is assumed that $z = x - x^*$. An acceptable range for z is obtained by convergence analysis of the series as follows:

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}z\right| = \lim_{n\to\infty}\frac{(2n-3)}{2(n+1)}\left|\frac{z}{x^*}\right| < 1. \tag{16}$$

So, we must have:

$$|z| < x^*. \tag{17}$$

It means that initial amplitude of motion must be smaller than x^*. This constraint guarantees that two bodies do not separate from each other during the motion.

Changing the variable from x to z, Eq. (13) becomes:

$$\ddot{z} + \lambda^2\sum_{n=1}^{N}\alpha_n z^n = 0, \tag{18}$$

in which α_n's are the coefficients of z^n in the Taylor series of $x^{1.5}$ about x^*:

$$\begin{aligned} &\alpha_1 = \frac{3}{2}(x^*)^{0.5}, \qquad \alpha_2 = \frac{3}{8}(x^*)^{-0.5} \\ &\alpha_3 = -\frac{1}{16}(x^*)^{-1.5}, \ldots. \end{aligned} \tag{19}$$

From the definition, x^* and F are related by the following equation:

$$F = m\lambda^2 (x^*)^{1.5}. \tag{20}$$

2.3. Nonlinear vibration: Lindstedt–Poincaré method

In order to implement the Lindstedt–Poincaré method to solve Eq. (18), the following series expansions, and the change of the variable from t to τ, are introduced [21]:

$$\begin{aligned} &z = \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \cdots, \\ &\tau = \omega t, \\ &\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \end{aligned} \tag{21}$$

Changing the variable from t to τ, and substituting z and ω from Eq. (21) into Eq. (18), we have:

$$\begin{aligned} &\left(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\right)^2 \frac{d^2}{d\tau^2}\left(\varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \cdots\right) \\ &\quad + \sum_{n=1}^{N}\bar{\alpha}_n\left(\varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + \cdots\right)^n = 0 \end{aligned} \tag{22}$$

in which $\bar{\alpha}_n$ is given by:

$$\bar{\alpha}_n = \lambda^2\alpha_n. \tag{23}$$

Equating the coefficients of $\varepsilon, \varepsilon^2, \ldots,$ to zero, results in:

$$\omega_0^2\frac{d^2}{d\tau^2}z_1 + \omega_0^2 z_1 = 0 \Rightarrow z_1'' + z_1 = 0, \tag{24}$$

$$\begin{aligned} &\varepsilon^2 : \omega_0^2\frac{d^2}{d\tau^2}z_2 + \omega_0^2 z_2 = -2\omega_0\omega_1\frac{d^2}{d\tau^2}z_1 - \bar{\alpha}_2 z_1^2 \\ &\varepsilon^3 : \omega_0^2\frac{d^2}{d\tau^2}z_3 + \omega_0^2 z_3 = -\left(\omega_1^2 + 2\omega_0\omega_2\right)\frac{d^2}{d\tau^2}z_1 \\ &\qquad - 2\omega_0\omega_1\frac{d^2}{d\tau^2}z_2 - 2\bar{\alpha}_1\bar{\alpha}_2 z_2 - \bar{\alpha}_3 z_1^3 \end{aligned} \tag{25}$$

$$\varepsilon^4 : \begin{cases} \omega_0^2\frac{d^2}{d\tau^2}z_4 + \omega_0^2 z_4 = -\left(2\omega_0\omega_3 + 2\omega_1\omega_2\right)\frac{d^2}{d\tau^2}z_1 \\ \quad - \left(\omega_1^2 + 2\omega_0\omega_2\right)\frac{d^2}{d\tau^2}z_2 - 2\omega_0\omega_1\frac{d^2}{d\tau^2}z_3 \\ \quad - \bar{\alpha}_2\left(z_2^2 + 2z_1z_3\right) - 3\bar{\alpha}_3 z_1^2 z_2 - \bar{\alpha}_4 z_1^4 \end{cases} \tag{26}$$

$$\varepsilon^5 : \begin{cases} \omega_0^2\frac{d^2}{d\tau^2}z_5 + \omega_0^2 z_5 = -\left(\omega_2^2 + 2\omega_1\omega_3 + 2\omega_0\omega_4\right) \\ \quad \times \frac{d^2}{d\tau^2}z_1 - \left(2\omega_0\omega_3 + 2\omega_1\omega_2\right)\frac{d^2}{d\tau^2}z_2 \\ \quad - \left(\omega_1^2 + 2\omega_0\omega_2\right)\frac{d^2}{d\tau^2}z_3 \\ \quad - 2\omega_0\omega_1\frac{d^2}{d\tau^2}z_4 - \bar{\alpha}_2\left(2z_1z_4 + 2z_2z_3\right) \\ \quad - \bar{\alpha}_3\left(3z_1^2z_3 + 3z_2^2z_1\right) - 4\bar{\alpha}_4 z_1^3 z_2 - \bar{\alpha}_5 z_1^5. \end{cases} \tag{27}$$

It is important to note that ω_0 is the natural frequency of the linearized system and is given by $\omega_0 = \sqrt{\bar{\alpha}_1}$.

Solving Eq. (24), z_1 is obtained as follows:

$$z_1 = a\cos(\varphi), \tag{28}$$

where $\varphi = \tau + \beta$ and a and β are constants. Substituting z_1 into the equations with higher powers of ε, and eliminating secular terms, we have [22]:

$$z_2 = -\frac{\bar{\alpha}_2 a^2}{2\omega_0^2}\left(1 - \frac{1}{3}\cos 2\varphi\right), \tag{29}$$

$$\omega_1 = 0, \tag{30}$$

$$\begin{aligned}
\omega_2 &= \frac{\left(9\bar{\alpha}_3\omega_0^2 - 10\bar{\alpha}_2^2\right)a^2}{24\omega_0^3},\\
\omega_3 &= 0,\\
z_3 &= \frac{a^3}{32\omega_0^2}\left(\frac{2\bar{\alpha}_2^2}{3\omega_0^2} + \bar{\alpha}_3\right)\cos 3\varphi,\\
z_4 &= \frac{1}{\omega_0^2}\left(B_0 - \frac{B_2}{3}\cos 2\varphi - \frac{B_4}{15}\cos 4\varphi\right),
\end{aligned} \tag{31}$$

where B_0, B_2 and B_4 are defined as:

$$\begin{aligned}
B_0 &= -\frac{19\bar{\alpha}_2^3 a^4}{72\omega_0^4} + \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{8\omega_0^2} - \frac{3\bar{\alpha}_4 a^4}{8},\\
B_2 &= -\frac{59\bar{\alpha}_2^3 a^4}{144\omega_0^4} + \frac{31\bar{\alpha}_3 a^4\bar{\alpha}_2}{32\omega_0^2} - \frac{\bar{\alpha}_4 a^4}{2},\\
B_4 &= -\frac{5\bar{\alpha}_2^3 a^4}{144\omega_0^4} - \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{32\omega_0^2} - \frac{\bar{\alpha}_4 a^4}{8}.
\end{aligned} \tag{32}$$

Substituting z_1 to z_4 into Eq. (27), ω_4 is evaluated as follows:

$$\omega_4 = -\frac{a^4}{\omega_0^7}\Bigg(\frac{7}{8}\bar{\alpha}_2\bar{\alpha}_4\omega_0^4 + \frac{15}{256}\bar{\alpha}_3^2\omega_0^4 - \frac{173}{192}\bar{\alpha}_3\omega_0^2\bar{\alpha}_2^2 + \frac{485}{1728}\bar{\alpha}_2^4 - \frac{5}{16}\bar{\alpha}_5\omega_0^6\Bigg). \tag{33}$$

So, ω and z are obtained as follows:

$$\omega = \omega_0 + \varepsilon^2\omega_2 + \varepsilon^4\omega_4 + O\left(\varepsilon^5\right), \tag{34}$$

$$\begin{aligned}
z = {} & -\frac{1}{2}\frac{\bar{\alpha}_2 a^2}{\omega_0^2} - \frac{19\bar{\alpha}_2^3 a^4}{72\omega_0^6} + \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{8\omega_0^4} - \frac{3\bar{\alpha}_4 a^4}{8\omega_0^2} + a\cos\varphi\\
& + \left\{\frac{\bar{\alpha}_2 a^2}{6\omega_0^2} + \frac{59\bar{\alpha}_2^3 a^4}{432\omega_0^6} + \frac{\bar{\alpha}_4 a^4}{6\omega_0^2} - \frac{31\bar{\alpha}_3 a^4\bar{\alpha}_2}{96\omega_0^4}\right\}\cos 2\varphi\\
& + \frac{1}{32}\frac{a^3\left(\frac{2\bar{\alpha}_2^2}{3\omega_0^2} + \bar{\alpha}_3\right)}{\omega_0^2}\cos 3\varphi\\
& + \frac{1}{15}\left\{\frac{\bar{\alpha}_4 a^4}{8\omega_0^2} + \frac{5\bar{\alpha}_2^3 a^4}{144\omega_0^6} + \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{32\omega_0^4}\right\}\cos 4\varphi.
\end{aligned} \tag{35}$$

The indeterminate parameters in the above equations are a and β, which are evaluated after applying the initial conditions. As a common initial condition, we assume:

$$\begin{cases} \dot{z}(0) = 0\\ z(0) = -z_0. \end{cases} \tag{36}$$

Applying $\dot{z}(0) = 0$ on Eq. (35), we have:

$$\begin{aligned}
& -a\sin\beta - 2\left\{\frac{\bar{\alpha}_2 a^2}{6\omega_0^2} + \frac{59\bar{\alpha}_2^3 a^4}{432\omega_0^6} + \frac{\bar{\alpha}_4 a^4}{6\omega_0^2} - \frac{31\bar{\alpha}_3 a^4\bar{\alpha}_2}{96\omega_0^4}\right\}\sin 2\beta\\
& + -\frac{3}{32}\frac{a^3\left(\frac{2\bar{\alpha}_2^2}{3\omega_0^2} + \bar{\alpha}_3\right)}{\omega_0^2}\sin 3\beta\\
& -\frac{4}{15}\left\{\frac{\bar{\alpha}_4 a^4}{8\omega_0^2} + \frac{5\bar{\alpha}_2^3 a^4}{144\omega_0^6} + \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{32\omega_0^4}\right\}\sin 4\beta = 0,
\end{aligned} \tag{37}$$

which gives $\beta = 0$ or $\beta = \pi$. Since $z(0) = -z_0$, $\beta = \pi$ is admissible. Substituting the value of β into Eq. (35), we have:

$$\begin{aligned}
& -\frac{1}{2}\frac{\bar{\alpha}_2 a^2}{\omega_0^2} - \frac{19\bar{\alpha}_2^3 a^4}{72\omega_0^6}\\
& + \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{8\omega_0^4} - \frac{3\bar{\alpha}_4 a^4}{8\omega_0^2} - a + \frac{\bar{\alpha}_2 a^2}{6\omega_0^2} + \frac{59\bar{\alpha}_2^3 a^4}{432\omega_0^6} + \frac{\bar{\alpha}_4 a^4}{6\omega_0^2}\\
& - \frac{31\bar{\alpha}_3 a^4\bar{\alpha}_2}{96\omega_0^4} - \frac{1}{32}\frac{a^3\left(\frac{2\bar{\alpha}_2^2}{3\omega_0^2} + \bar{\alpha}_3\right)}{\omega_0^2}\\
& + \frac{1}{15}\left\{\frac{\bar{\alpha}_4 a^4}{8\omega_0^2} + \frac{5\bar{\alpha}_2^3 a^4}{144\omega_0^6} + \frac{5\bar{\alpha}_3 a^4\bar{\alpha}_2}{32\omega_0^4}\right\} = -z_0.
\end{aligned} \tag{38}$$

In order to find a, we first rewrite equation (38) into the form of:

$$g(a) = A_4 a^4 + A_3 a^3 + A_2 a^2 + A_1 a + z_0 = 0, \tag{39}$$

where A_n's are the coefficients of a^n's evaluated from Eq. (38).

According to Eq. (28) in the linear approximation, a is equal to the initial amplitude (z_0), but when higher approximations are used, it is not exactly equal to z_0 and must be calculated from Eq. (39).

In order to solve Eq. (39), the Newton–Raphson method is used. If $\bar{a}_n$ is the root value obtained in step n, we have:

$$\bar{a}_{n+1} = \bar{a}_n - \frac{g(\bar{a}_n)}{g'(\bar{a}_n)}. \tag{40}$$

It is important to note that the initial guess for the solution is chosen as $\bar{a}_0 = z_0$.

By evaluating a and substituting it into Eq. (35), the dynamic response of the system and its natural frequency are obtained.

3. Results and discussion

In order to present results, we first calculate necessary parameters. For this purpose, we assume that the ball and plate are made of steel [23].

$$\begin{aligned}
& E_1 = E_2 = 200\ \text{GPa},\\
& \nu_1 = \nu_2 = 0.28,\\
& d_1 = 10\ \text{mm},\\
& d_2 = \infty.
\end{aligned} \tag{41}$$

Since holding two spheres together in vertical position is practically very difficult and maybe impossible, the value of d_2 is chosen as infinity, in order to be easily experimented in future work.

By determining mechanical and geometrical properties, K_a is evaluated from Eq. (6) as follows:

$$K_a = 3.25 \times 10^{-5}. \tag{42}$$

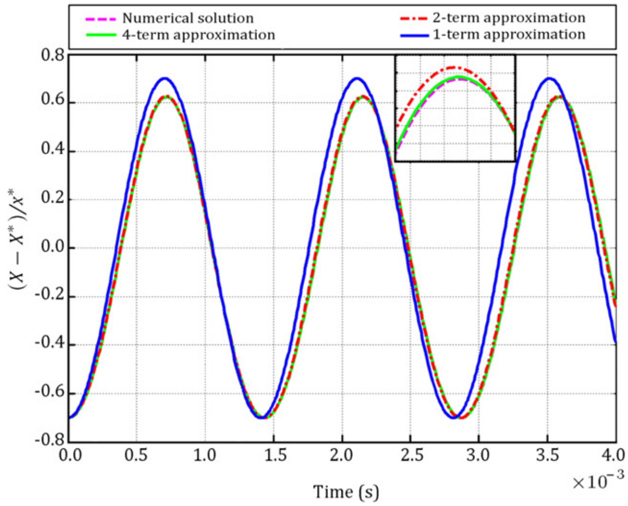

Figure 3: Time response of the center of the sphere ($z_0 = 0.7X^*$).

In order to determine the contact force, we assume that the maximum pressure in the contact area of the sphere is equal to 1200 MPa, which is noticeably smaller than the yield stress of the used steel ($S_y = 2000$ MPa [23]). So, we have:

$$F = 19\ \text{N}. \tag{43}$$

It is highly important to note that this force is resulted due to the mass of other components supported by the bearing (or the wheel), therefore, we have considered its inertia effect in the equations. It means that the mass value is considered as:

$$m = \frac{F}{9.81} = 1.93\ \text{kg}. \tag{44}$$

Substituting the contact force into Eq. (5), the value of a is obtained. So, the value of C appearing in Eq. (9) can be evaluated as follows:

$$C = \left| \int_0^{R/a} \left(\frac{-1}{1+\zeta_a^2} \right) d\zeta_a + 2\nu \int_0^{R/a} \left(\left[1 - |\zeta_a| \tan^{-1} \times \left(\frac{1}{|\zeta_a|} \right) \right] (1+\nu) - \frac{1}{2\left(1+\zeta_a^2\right)} \right) d\zeta_a \right| = 1.41. \tag{45}$$

The initial deflection of the center of the sphere resulted by the compressible force is:

$$x^* = (F)^{2/3} \left(\frac{3C}{2\pi E K_a} \right) = 0.73\ \mu\text{m}. \tag{46}$$

After evaluating all necessary parameters, Eq. (12) can be solved.

In Figure 3, the dynamic motion of the center of the sphere, with initial conditions as mentioned in Eq. (36), is presented. Different orders of approximation have been used and compared with the numerical solution obtained by the RK45 solver in MATLAB®. It is obvious that the linear solution has a large error in comparison with other approximations.

It is highly important to note that the difference between numerical solution and 4-term approximation is less than 1%, which shows good agreement between the two methods.

It can be seen in Figure 3 that minimum and maximum displacements from static equilibrium are not equal. This indicates that the motion amplitude in compression differs from the amplitude in a reverse direction. This phenomenon can be justified by the nonlinearity of the system.

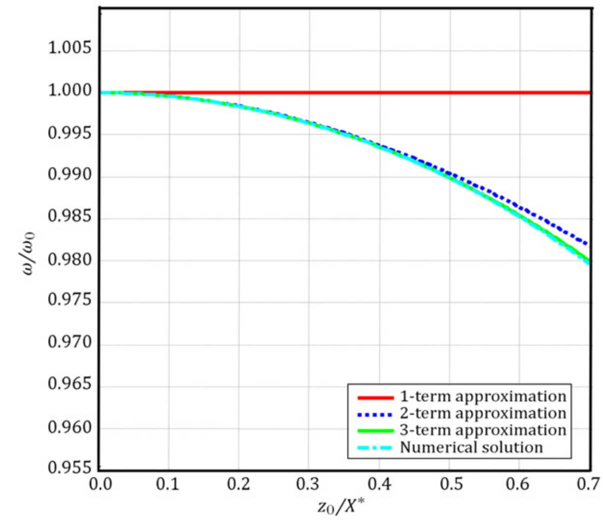

Figure 4: Diagram of frequency ratio vs. ratio of initial amplitude to static deflection ($X^* = 0.73\ \mu$m, $\omega 0 = 4467$ rad/s).

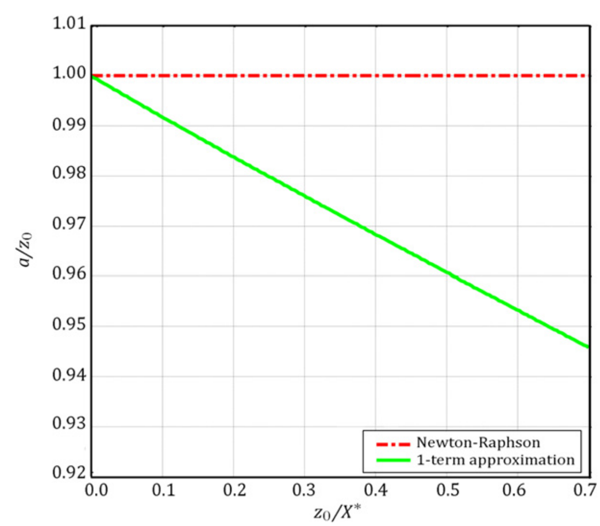

Figure 5: Diagram of amplitude ratio vs. ratio of initial amplitude to static deflection ($X^* = 0.73\ \mu$m).

In Figure 4, the dependency of frequency on the initial amplitude of vibration has been indicated, and a comparison between different orders of approximation and numerical methods (RK45 solver of MATLAB®) has been done. It is highly important to note that ω_0 is the natural frequency of the linearized system. It is evident that in the linear solution, the frequency is independent of the initial amplitude, and the frequency ratio equals unity.

It was discussed previously that a is an indicator of amplitude. Also, it was stated that for large values of initial amplitude (z_0), the value obtained for a differs from z_0. This concept is indicated in Figure 5. As the initial amplitude of the motion is increased, the difference between a and z_0 is increased, too.

In Figure 6, variation of force versus displacement from the equilibrium position is presented using different orders of approximation. It can be seen that the force predicted by all approximations is equal in zero displacement, because Taylor expansion has been obtained about this point. It can be

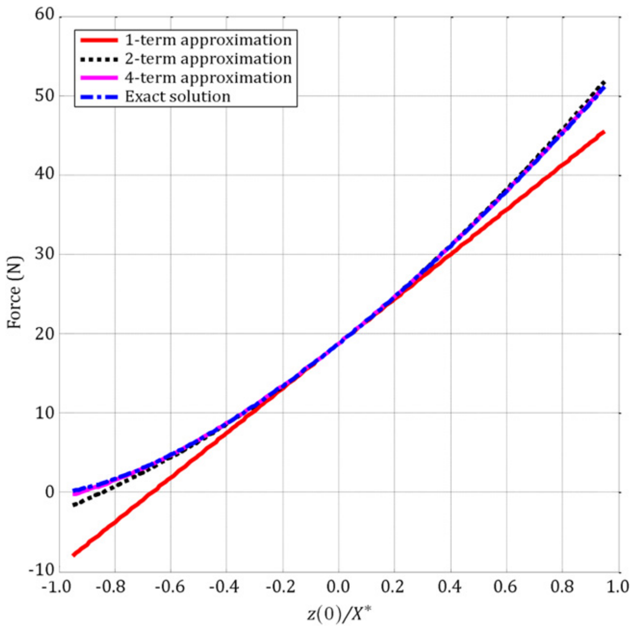

Figure 6: Force vs. displacement from static equilibrium position ($X^* = 0.73$ μm).

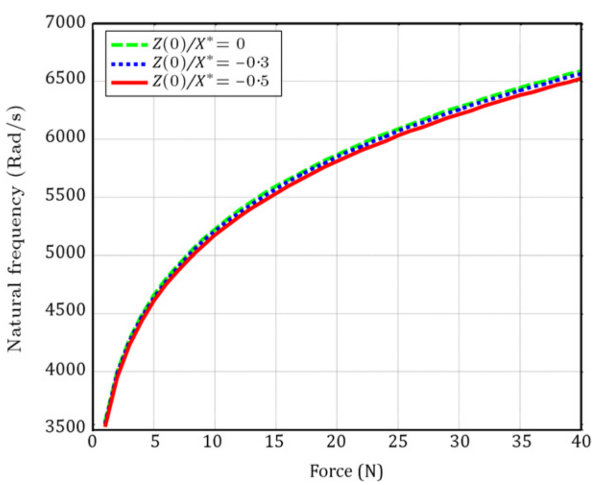

Figure 7: Dependency of the frequency on the preload.

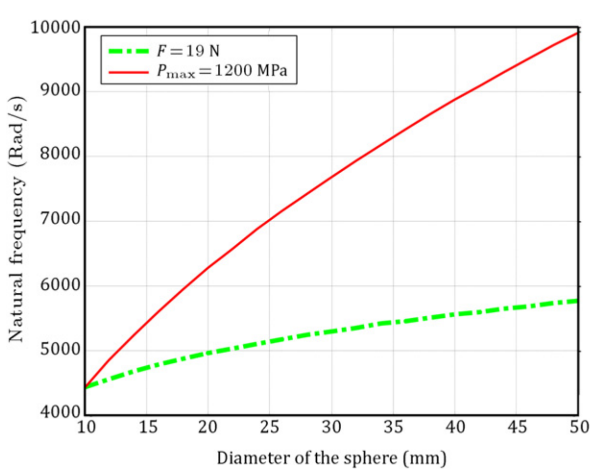

Figure 8: Frequency vs. diameter of the sphere ($z_0 = 0.5X^*$).

recognized that the graph slope increases with the vibration amplitude. So, by increasing the compressive force, the stiffness of the system increases, too.

Dependency of frequency on the preload applied to the ball has been illustrated in Figure 7. It can be seen that by increasing the preload, natural frequency is increased. Also, three different values are set for the initial amplitude ($\frac{z(0)}{X^*} = 0, -0.3, -0.5$) to illustrate the effect of initial amplitude on frequency. It is obvious that by increasing initial amplitude, the frequency is decreased.

In Figure 8, frequency is presented versus ball diameter. The effect of diameter is studied in two different cases: the condition that P_{max} (the maximum pressure due to the preload) is constant (solid line) and the condition that preload is constant (dashed line). It can be seen that by increasing the diameter of the ball, the frequency is increased.

The very important point concluded from Figures 7 and 8 is that decreasing diameter and preload both decrease natural frequency. This effect must be considered in the design to put the working frequency away from the natural frequency of the system.

4. Summary and conclusion

In this paper, oscillation of a spherical ball on a surface under a compressive force was studied. The Hertzian contact theory was used to obtain stress components in the sphere. By implementing the Lindstedt–Poincaré method, frequency of vibration and the dynamic motion of the center of the sphere were obtained. Comparisons show that the results of 4-term approximation are very close to those extracted by the RK45 solver of MATLAB®. Also, results indicate that the linear solution has a significant error, especially in large deflections.

The dependency of vibration frequency on several parameters, such as preload, diameter of sphere and initial amplitude, must be considered in the design to prevent a possible overlap of the natural frequency of the system with the working frequency, especially when the magnitude of the force varies in a wide range. This can be more critical for balls made of flexible materials or with small diameters.

References

[1] Yamamoto, H. "On the vibration of a shaft supported by bearing having radial clearances", *Trans. JSME*, 21, pp. 182–192 (1955).
[2] Gustafson, O.G. and Tallian, T. "Research report on study of the vibration characteristics of bearings", Reg: 58514: 4223, December, Report: AL 631023, SKF Ind. Inc. (1963).
[3] Meyer, L.D., et al. "An analytical model for ball bearing vibrations to predict vibration response to distributed defects", *Trans. ASME J. Mech. Des.*, 102, pp. 205–210 (1980).
[4] Ehrich, F.F. and O'Connor, J.J. "Stator whirl rotors in bearing clearance", *Trans. ASME J. Eng. Ind.*, pp. 381–390 (1967).
[5] Ehrich, F.F. "High order subharmonic response if high speed rotors in bearing clearance", *Trans. ASME J. Vib. Acoust. Stress Reliab. Des.*, 110, pp. 9–16 (1988).
[6] Choi, Y.S. and Noah, S.T. "Nonlinear steady state response of a rotor-support system", *Trans. ASME J. Vib. Acoust. Stress Reliab. Des.*, 109, pp. 255–261 (1987).
[7] Tiwari, M., et al. "Effect of a ball bearing on the dynamics of a balanced horizontal rotor", *J. Sound Vib.*, 238(5), pp. 723–756 (2000).
[8] Harsha, S.P., et al. "Non-linear dynamic behaviors of rolling element bearings due to surface waviness", *J. Sound Vib.*, 272, pp. 557–580 (2004).
[9] Nataraj, C. and Harsha, S.P. "The effect of bearing cage run-out on the nonlinear dynamics of a rotation shaft", *Commun. Nonlinear Sci. Numer. Simul.*, 13, pp. 822–838 (2008).
[10] Sinou, J.J. "Non-linear dynamics and contacts of an unbalanced flexible rotor supported on ball bearings", *Mech. Mach. Theory*, 44, pp. 1713–1732 (2009).
[11] Nielsen, J.C.O. and Johansson, A. "Out-of-round railway wheels; a literature survey", *Proc. Inst. Mech. Eng. F*, 214, pp. 79–91 (2000).

[12] Nielsen, J.C.O. and Igeland, A. "Vertical dynamic interaction between train and track; influence of wheel and track imperfections", *J. Sound Vib.*, 187, pp. 825–839 (1995).
[13] Grassie, S.L. "Rail corrugation: characteristics, causes, and treatments", *Proc. Inst. Mech. Eng. Part F*, 223, pp. 581–596 (2009).
[14] Li, Z., et al. "An investigation into the causes of squats-correlation analysis and numerical modeling", *Wear*, 265, pp. 1349–1355 (2008).
[15] Thompson, D., *Railway Noise and Vibration: Mechanisms, Modeling and Means of Control*, Elsevier ISBN: 978-0-08-045147-3 (2009).
[16] Thompson, D.J. "Wheel–rail noise generation, part III: rail vibration", *J. Sound Vib.*, 161, pp. 421–446 (1993).
[17] Yan, W. and Fischer, F.D. "Applicability of the Hertz contact theory to rail-wheel contact problems", *Arch. Appl. Mech.*, 70, pp. 255–268 (2000).
[18] Nielsen, J.B. "Evolution of rail corrugation predicted with a non-linear wear model", *J. Sound Vib.*, 227(5), pp. 915–933 (1999).
[19] Baeza, L., et al. "Railway train-track dynamics for wheelflats with improved contact models", *Non-Linear Dynam.*, 45, pp. 385–397 (2006).
[20] El-Sayed, H.R. "Stiffness of deep-groove ball bearings", *Wear*, 63, pp. 89–94 (1980).
[21] Shigley, J.E. and Mischke, C.R. "Load and stress analysis", In *Mechanical Engineering Design*, 5th Edn., pp. 161–165, McGraw Hill, New York, USA (1989).
[22] Nayfeh, A.H. and Mook, D.T. "Conservative single-degree-of-freedom systems", In *Nonlinear Oscillations*, 1st Edn., pp. 39–77, John Wiley & Sons, Inc., New York, USA (1979).
[23] Harris, T.A. and Kotzalas, M.N. "Structural materials of bearings", In *Essential Concepts of Bearing Technology*, 5th Edn., pp. 296–348, Taylor & Francis Group, New York, USA (2007).

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